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Baryogenesis, Dark Matter and the Pentagon

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Abstract: We present a new mechanism for baryogenesis, which links the baryon asymmetry of the universe to the dark matter density. The mechanism arises naturally in the Pentagon model of TeV scale physics. In that context, it forces a re-evaluation of some of the assumptions of the model, and we detail the changes that are required in order to fit observations.

Contents

1. Introduction

With a few exceptions[1] most models of the early universe make no attempt to connect the observed baryon and dark matter densities. Dark matter is usually assumed to consist of neutralinos or axions, and there is no connection between the properties of these particles, and the baryon asymmetry. In this paper we will introduce a new class of models in which the solutions to these two problems are directly related. These models are motivated by the Pentagon model of TeV scale physics[2], where the mechanism we will discuss arises naturally. Indeed, it is forced on us if we insist on finding a dark matter candidate within the low energy model itself.

The basic new structure in this class of models is an approximate symmetry, with current J^{μ}_{PB} . This symmetry is explicitly broken and an asymmetry ϵ for J^0_{PB} is generated in the very early universe. The leading symmetry violating operator at low energy has dimension D and comes from physics at a scale M_b. The symmetry is also spontaneously broken at a scale f. Finally, there is a coupling in the low energy effective Lagrangian of the form

$$\frac{1}{\Lambda^2} J^{\mu}_{PB} B_{\mu},$$

where B_{μ} is the ordinary baryon number current.

The asymmetry in J_{PB} then acts as an effective chemical potential for ordinary baryon number. Electroweak sphaleron processes will then generate a baryon asymmetry (and a compensating lepton asymmetry). This is a form of spontaneous baryogenesis, a mechanism invented by Cohen and Kaplan[3].

At a lower energy scale, the effects of spontaneous and explicit violation of J_{PB} set in. These convert the energy stored initially in the asymmetry into a condensate of pseudo-Goldstone bosons, which can be the dark matter. We can use the parameters of the model to fit both the baryon asymmetry, and dark matter density, in a variety of ways. However, in the version of the model that arises from the Pentagon, all parameters but ϵ and M_b are determined in terms of other experimental quantities. We find that, given the assumption in [2] that M_b is at least as large as the neutrino seesaw scale (5×10^{14} GeV)the baryon asymmetry is too large, or the dark matter density too small. We describe modifications of the model which remove this problem in section 4.

2. Pseudo-goldstone dark matter with an asymmetry

The idea of pseudo-Nambu-Goldstone bosons as dark matter is familiar from axion models. The general idea is that, after inflation, the pseudo Goldstone field has a fixed value and small velocity. Its motion is friction dominated until the Hubble parameter falls to the axion mass, at which time it begins to oscillate and behaves like cold dark matter. For appropriate values of the mass of the field and its decay constant, a PNGB can reproduce the observed dark matter density.

Here we propose an alternative initial condition for PNGB dark matter. The symmetry associated with the PNGB is explicitly broken. We assume that this breaking is larger during inflation than it is at low energies. A large value for the inflaton field, typical in slow roll inflation models, can enhance operators that are highly irrelevant at low energies. We assume that this produces an asymmetry ϵ at the end of inflation, and that there is not much entropy production between the time this asymmetry is generated, and the present. Note that the Affleck-Dine mechanism can easily generate large values of ϵ[5].

We note parenthetically that in string theory most PNGB's arise as duals of fundamental anti-symmetric tensor gauge fields. From this point of view, the assumption of an asymmetry is equivalent to assuming an expectation value for the magnetic field strength H_{ijk} and has been recently studied by Ibanez[4]. Our work will be primarily devoted to PNGBs of accidental symmetries of low energy gauge fields, but for completeness we will also study what happens to a QCD axion with asymmetric initial conditions.

The non-zero value of ϵ and our assumption of isentropic cosmological expansion allows us to write an equation for the time dependence of the density J^0_{PB}

$$J^0_{PB} = \epsilon g T^3, \tag{2.1}$$

where g is the effective number of massless degrees of freedom. This equation is valid from the moment of asymmetry generation, until some time at which low energy J^0_{PB} violation becomes important, or a large amount of entropy is dumped into the universe. Note that it is valid both above and below the scale of *spontaneous* breaking of the approximate symmetry. Below that scale, which we denote by f, we have

$$J^0_{PB} = f\partial_0 b, \tag{2.2}$$

where b is the PNGB field. The primary focus of this paper is on models in which f is dynamically generated by strong gauge dynamics.

Assume that the explicit breaking of the symmetry after inflation comes primarily from an operator of dimension D, and has its origin in physics which decouples at the scale M_b. Below the scale f this will give rise to an effective potential for b

$$\frac{\Lambda_L^D}{M_b^{D-4}} V(b/f). \tag{2.3}$$

For a QCD axion we would have $D = 4$ and $\Lambda_L = \Lambda_{QCD}$. The cross over temperature, at which this potential begins to affect the evolution of the b field is given by

$$\frac{(\epsilon g (T*)^3)^2}{2} = \frac{f^2 \Lambda_L^D}{M_b^{D-4}}, \tag{2.4}$$

or

$$\epsilon g (T*)^3 = \sqrt{2} f M_b^2 \frac{\Lambda_L}{M_b}^{D/2} \tag{2.5}$$

At this temperature, the ratio of energy densities of dark matter to radiation is:

$$\left[\frac{\rho_b}{\rho_\gamma}\right]_{T*} = \frac{(\epsilon g (T*)^3)^2}{f^2 g (T*)^4}. \tag{2.6}$$

Below this temperature, the b field begins to oscillate, and its energy scales like that of cold dark matter. The dark matter to radiation ratio at any lower temperature is

$$\frac{\rho_b}{\rho_\gamma} = \frac{(\epsilon g (T*)^3)^2}{f^2 g (T*)^4} \frac{T*}{T}. \tag{2.7}$$

An asymmetric PNGB can be a good dark matter candidate if this ratio becomes one at matter radiation equality, $T_{eq} \sim 1\text{eV}$. Using the equation for $T*$, this requires

$$M_b^{\frac{D}{2}-2} = \frac{\epsilon}{f T_{eq}} \Lambda_L^{D/2}. \tag{2.8}$$

3. Spontaneous baryogenesis

Suppose in addition that there is a coupling

$$\frac{1}{\Lambda^2} J_{PB}^\mu B_\mu, \tag{3.1}$$

between the current of the PNGB and the ordinary baryon number current. Assume for the moment that $T* < T_{sh}$, where T_{sh} is the scale below which the electroweak baryon violating process shuts off, with $T_{sh} \sim 100$ GeV. Then, in the regime where electroweak baryon violation is in equilibrium, this coupling has the form

$$\epsilon g \frac{T^3}{\Lambda^2} B_0 \equiv \tilde{\mu} B_0, \tag{3.2}$$

which is a time dependent chemical potential for baryon number. The relative rate of change of the chemical potential is the expansion rate of the universe, much slower than sphaleron processes. Thus, the combination of $\tilde{\mu}$ and the sphaleron process put us in the regime of equilibrium spontaneous baryogenesis, as defined by Cohen and Kaplan[3]. Other baryon violating processes could be hypothesized, at a variety of energy scales. Their contribution to the total baryon asymmetry would add to the one we compute here. Without fine tuning, it is unlikely that the contributions of other processes could cancel the electroweak sphaleron induced asymmetry. It is straightforward to compute the induced baryon asymmetry (see Appendix). We will also assume that any chemical potential for lepton number (recall that $B-L$ is conserved by sphaleron processes) is much smaller than that for baryon number, and that $B-L$ asymmetries generated in the early universe are small compared to the baryon asymmetry generated by our mechanism. We find

$$\epsilon_B = \frac{< B >}{gVT^3} = \frac{1}{2}\epsilon(\frac{T_{sh}}{\Lambda})^2 = 6 \times 10^{-10}. \tag{3.3}$$

The last equality is the constraint from the observed baryon asymmetry.

Note that if $T*$ is substantially larger than T_{sh} then no asymmetry is generated because the chemical potential is turned off before baryon number violating processes go out of equilibrium. The asymmetry will equilibrate to whatever value of the $B-L$ asymmetry was generated by early universe processes like leptogenesis.

Let us plug this value of ϵ into the formula we got by insisting that the dark matter density comes out right. then we get

$$M_b^{\frac{D}{2}-2} = 1.2 \times 10^{-9} \frac{\Lambda^2}{f\ T_{sh}^2 T_{eq}} \Lambda_L^{D/2}. \tag{3.4}$$

The condition that $T* < T_{sh}$ translates into the inequality

$$\Lambda_L^D < g M_b^{D-4} T_{sh}^3 T_{eq}, \tag{3.5}$$

or

$$\left(\frac{M_b}{\Lambda_L}\right)^{D/4} > \frac{M_b}{g^{1/4} 0.18 GeV} \tag{3.6}$$

while the plausible constraint that $\epsilon < 10^3$ is

$$1.2(\frac{\Lambda}{T_{sh}})^2 < 10^{12}. \tag{3.7}$$

For a QCD axion we expect $\Lambda > f$, where a strictly greater than sign is used because the coupling between the axion field and the baryon number current violates CP and might be suppressed by more than dimensional analysis. The conventional lower bounds on the axion decay constant, from red giants and supernovae, then rule out this kind of asymmetric axion scenario. Note also that such a scenario would have required a value of ϵ which is probably too large to be generated by the Affleck-Dine mechanism.

Our analysis was based on the assumption that the pseudo-Goldstone boson was the correct description of physics at the scale where the primordial J^0_{PB} asymmetry is wiped out by processes which violate this symmetry. The temperature $T*$ where this occurs must thus be smaller than f. Scenarios where this inequality is not satisfied are more complicated to analyze. Some of them could give rise to acceptable cosmologies. However, both the QCD axion models, and the Pentagon model, satisfy $T* < f$ so we will not attempt to analyze this possibility any further.

4. Spontaneous baryogenesis and dark matter in the Pentagon model

In the Pentagon model, all of the previous ingredients are present, and most of the parameters are related. There is a spontaneously broken accidental symmetry, penta-baryon number. It has $f = y$ TeV and $\Lambda_5 = x$ TeV, where x and y are of order $1 - 10$ and most probably $y > x$ Using various different estimates of standard model superpartner masses that have appeared in the literature, we find x running between 1.5 and 7. There is also a current current coupling to baryon number with $\Lambda_5 = \alpha_3 \Lambda$, where the strong coupling is evaluated at the TeV scale and is $\sim .1$.

The lowest dimension penta-baryon number violating operators, which preserve gauge invariance, SUSY, and the fundamental discrete R symmetry of the model are

$$\int d^2\theta \; SP^5, \tag{4.1}$$

and

$$\int d^2\theta \; S\tilde{P}^5. \tag{4.2}$$

These have dimension $D = 7$[1]. In [2] TB also imposed the natural constraint that all allowed irrelevant operators were suppressed by powers of $M_U \sim 10^{15}$ GeV, which is the scale that appears in the neutrino seesaw term. Let us let the scale in the dimension 7 operator be a free parameter, M_b.

The requirement that we get the right baryon asymmetry is

$$\epsilon_B = \frac{1}{2}\epsilon(\frac{\alpha_3 T_{sh}}{\Lambda_5})^2, \tag{4.3}$$

[1]In [2] TB forgot the R symmetry constraint, and used the $D = 6$ operators without the singlet S.

or

$$\epsilon = 1.2x^2 \times 10^{-5}. \tag{4.4}$$

The equation determining $T*$ is

$$\frac{(\epsilon g(T*)^3)^2}{2f^2} = \frac{\Lambda_5^7}{M_b^3}, \tag{4.5}$$

or

$$\epsilon g(T*)^3 = \sqrt{2} f \frac{{\Lambda_5}^{7/2}}{M_b^{3/2}}. \tag{4.6}$$

According to WMAP, the temperature of matter radiation equality is about 1 eV. Thus, the condition that the penton is dark matter is

$$1 = \frac{\rho_b}{\rho_\gamma} = \frac{(\epsilon g(T*)^3)^2}{f^2 g(T*)^4} \frac{T*}{T_{eq}}. \tag{4.7}$$

If we insert the values for ϵ and T_{eq} into this equation, we get the constraint

$$M_b = (\frac{x}{2y})^{2/3} x^3 \times 10^8 \text{GeV}. \tag{4.8}$$

The prefactor is a number slightly less than 1, so this ranges between about $3 \times 10^8 \rightarrow 3 \times 10^{10}$ GeV as x ranges over the values allowed by the various estimates of superpartner masses. We also note that over the whole range of x, the inequality $T* < T_{sh}$ is satisfied, as long as y/x is not too large.

The low scale M_b raises the specter of unacceptably fast proton decay. There is a dimension 6 operator of the schematic form

$$\frac{1}{M^2} \int d^2\theta Q^3 L S,$$

which is invariant under all of the symmetries of the Pentagon model. With $M = M_b$ this would lead to disaster. Thus, if we want to use the penton to generate the dark matter in the universe, we must construct a theory at the scale M_b which explains the absence of operators of dimension 6, which could contribute to proton decay.

The alternative is to abandon the penton field b as the origin of dark matter, which is to say that ϵ is presumed to be small. We could retain a non-zero value of ϵ as the mechanism for baryogenesis, though this would be no more attractive than a host of other options. Dark matter would have to come from somewhere outside the Pentagon. A possible candidate is an axion dual to an antisymmetric tensor field. This could also solve the strong CP problem and be compatible with estimates from string theory[7] .

Yet a third alternative, is to raise the values of Λ_5 and m_{ISS}, probably abandoning the hypothetical connection of the model to Cosmological SUSY Breaking. A value of $\Lambda_5 \sim 3 \times 10^5$ GeV and $M_b = 10^{15}$ GeV seems to be compatible with both the observed dark matter density and baryon asymmetry. The value of ϵ is a bit less than 1 and $T*$ is well below the sphaleron mass. However, there are a number of phenomenological problems with this suggestion. The rough estimates for sparticle masses and the electroweak scale in the Pentagon model are

$$m_{1/2}^{(i)} \sim \frac{50}{3} g_S \frac{\alpha_i}{A\pi} m_{ISS}, \tag{4.9}$$

$$m_{\tilde{e}_R} \sim \sqrt{\frac{50}{3} \frac{\alpha_1}{B\pi}} m_{ISS}, \tag{4.10}$$

$$H_u \sim 240\text{GeV} \sim \frac{g_S}{6} \Lambda_5. \tag{4.11}$$

$$m_{\tilde{q}} \sim \sqrt{\frac{50}{3} \frac{\alpha_3}{B\pi}} m_{ISS}. \tag{4.12}$$

m_{ISS} is the mass term which creates a meta-stable SUSY violating vacuum in the Pentagon model. The value of A runs between 1 and 8, while that of B is between 1 and 4. g_S is a Yukawa coupling between a next-to-minimal SSM singlet, and the Pentagon fields. Other squark and slepton masses are larger than that of the right handed selectron by factors of $\frac{\alpha_{2,3}}{\alpha_1}$.

The factor of 1/6 in the equation for the electroweak scale represents a pious hope that the original Pentagon model with $\Lambda_5 \sim 1.5$ TeV does not suffer from a *little hierarchy problem*. That is, the dimensional analysis estimate of the electroweak scale has 1/6 replaced by 1, and we have to hope that dynamical calculations in the strongly coupled Pentagon model provide the factor of 6.

If we postulate $\Lambda_5 \sim 3 \times 10^5$ GeV, dynamical suppression is no longer plausible. We can get the correct electroweak scale by choosing g_S small, about 5×10^{-3}, but this implies small gaugino masses. The ratio between squark and wino masses is

$$\frac{m_{\tilde{q}}}{m_{\tilde{w}}} \sim \frac{A}{2B} \frac{\alpha_3}{\alpha_2} 10^2. \tag{4.13}$$

Using the experimental lower bound on the wino mass we get squark masses that are a few times 10 TeV . The model then has a hierarchy problem, and radiative corrections to the Higgs mass are substantially larger than the values indicated by precision electroweak fits.

Even a value $M_b = 10^{15}$ GeV is not enough to protect us from proton decay. The unified values of gauge couplings are quite large in the Pentagon model, so even dimension six proton decay operators must be suppressed by 10^{16} GeV or so. We conclude that raising the scale Λ_5 does not seem to be a promising avenue for making a working model of penton dark matter and baryogenesis.

5. Conclusions

The Pentagon model suggests a novel form of spontaneous baryogenesis, which can tie together the dark matter density and the baryon asymmetry. While the general idea works quite well, it does not work in the Pentagon model, unless we contemplate a scale of $10^8 \rightarrow 10^{10}$ GeV for the leading irrelevant operator that violates penta-baryon number. It remains to be seen whether one can invent a high energy extension of the model to generate this operator without generating standard model operators already ruled out by experiment.

If we stick with 10^{15} GeV as the scale for all irrelevant corrections to the Lagrangian, then we must abandon the penton theory of dark matter. The penton is a light PNGB which might be produced in the laboratory, but its cosmological abundance is negligible. It could still participate in the generation of the baryon asymmetry we observe.

The most plausible candidate for dark matter compatible with the Pentagon would then be a QCD axion. This would also solve the strong CP problem.

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7. Appendix

In this appendix we go through the calculation of the baryon number expectation, in the presence of a chemical potential in thermal equilibrium. We begin by ignoring leptons, which would play a role in determining the equilibrium baryon number generated through processes such as the electroweak sphaleron where the quantity $B - L$ is conserved. After computing $\langle B \rangle$ first by ignoring this constraint, we will then modify our calculation to include it. This is mostly for pedagogical purposes, but the naive calculation turns out to give the right order of magnitude as long as the lepton chemical potential is of equal or smaller magnitude than the baryon chemical potential.

Statistically, the equilibrium value of B can be calculated from the partition function:

$$\langle B \rangle = \frac{\partial}{\partial \mu} \ln Z$$

The μ being used here is dimensionless. In terms of the usual chemical potential $\tilde{\mu}$, which has dimensions of energy, $\mu = \frac{\tilde{\mu}}{T} = \tilde{\mu}\beta$.

The partition function for the baryons including the chemical potential is

$$Z = Tr(e^{-\beta H + \mu \int d^4x q_i^\dagger q_i})$$

where

$$H = \sum_{i,k} |k|[a_k^{(i)\dagger} a_k^{(i)} + b_k^{(i)\dagger} b_k^{(i)}]$$

$$\int d^4x \, q^\dagger q = \frac{1}{3} \sum_k [a_k^\dagger a_k - b_k^\dagger b_k]$$

This can be re-expressed as

$$Z = Tr[\exp(-\sum_{i,k} (\beta|k| - \frac{\mu}{3}) a_k^{(i)\dagger} a_k^{(i)}) \exp(-\sum_{i,k} (\beta|k| + \frac{\mu}{3}) b_k^{(i)\dagger} b_k^{(i)})]$$

$$= Tr[\prod_{i,k} \exp(-(\beta|k| - \frac{\mu}{3}) a_k^{(i)\dagger} a_k^{(i)}) \exp(-(\beta|k| + \frac{\mu}{3}) b_k^{(i)\dagger} b_k^{(i)})]$$

The trace gives a sum over all possible states of the system; for fermions, each state can only have occupation number equal to zero or one, thus

$$Z = \prod_{k,i} (1 + e^{-(\beta|k| - \frac{\mu}{3})})(1 + e^{-(\beta|k| + \frac{\mu}{3})})$$

We are interested in the logarithm of Z, which allows us to express the logarithm of the product as a sum of logarithms

$$\ln Z = 6 \times 3 \times 2 \times \sum_k [\ln(1 + e^{-(\beta|k| + \frac{\mu}{3})}) + \ln(1 + e^{-(\beta|k| - \frac{\mu}{3})})]$$

The factor of $6 \times 3 \times 2$ comes from the sum over i which includes six flavors of quarks, three colors, and two spin states. Taking k to the continuum limit we have

$$\sum_k \rightarrow V \int \frac{d^3k}{(2\pi)^3}$$

and taking the μ derivative gives

$$\langle B\rangle = \frac{\partial}{\partial\mu}\ln Z = 36V\int\frac{d^3k}{(2\pi)^3}[\frac{1}{3(e^{\beta(k+\frac{\mu}{3\beta})})} - \frac{1}{3(e^{\beta(k-\frac{\mu}{3\beta})})}]$$

Using spherical coordinates, the angular integral just gives a factor of 4π, which leaves an integral over the magnitude k:

$$\langle B\rangle = \frac{6V}{\pi^2}(I_1 - I_2)$$

where

$$I_1 = \int_0^\infty dk\frac{k^2}{e^{\beta(k-\frac{\mu}{3\beta})}+1}$$

$$I_2 = \int_0^\infty dk\frac{k^2}{e^{\beta(k+\frac{\mu}{3\beta})}+1}$$

By substituting $x = \beta(k - \frac{\mu}{3\beta})$, I_1 becomes:

$$I_1 = \frac{1}{\beta^3}\int_{-\frac{\mu}{3}}^{\infty} dx\frac{(x+\frac{\mu}{3})^2}{e^x+1} = \frac{1}{\beta^3}\int_0^\infty dx\frac{(x+\frac{\mu}{3})^2}{e^x+1} + \frac{1}{\beta^3}\int_{-\frac{\mu}{3}}^{0} dx\frac{(x+\frac{\mu}{3})^2}{e^x+1}$$

and by substituting $x = \beta(k + \frac{\mu}{3\beta})$, I_2 becomes:

$$I_2 = \frac{1}{\beta^3}\int_{\frac{\mu}{3}}^{\infty} dx\frac{(x-\frac{\mu}{3})^2}{e^x+1} = \frac{1}{\beta^3}\int_0^\infty dx\frac{(x-\frac{\mu}{3})^2}{e^x+1} - \frac{1}{\beta^3}\int_0^{\frac{\mu}{3}} dx\frac{(x-\frac{\mu}{3})^2}{e^x+1}$$

The two can then be recombined into:

$$I_1 - I_2 = \frac{1}{\beta^3}\Big[\int_0^\infty dx\frac{4x(\frac{\mu}{3})}{e^x+1} + \int_{-\frac{\mu}{3}}^{0} dx\frac{(x+\frac{\mu}{3})^2}{e^x+1} + \int_{\frac{\mu}{3}}^{0} dx\frac{(x-\frac{\mu}{3})^2}{e^x+1}\Big]$$

$$I_1 - I_2 = \frac{1}{\beta^3}\Big[(\frac{4\mu}{3})(\frac{\pi^2}{12}) + \int_0^{\frac{\mu}{3}} dx\{\frac{(x-\frac{\mu}{3})^2}{e^{-x}+1} + \frac{(x-\frac{\mu}{3})^2}{e^x+1}\}\Big]$$

Miraculously, the exponentials simply add up to 1, leaving an integral of a polynomial:

$$= \frac{1}{\beta^3}\Big[\frac{\pi^2\mu}{9} + \int_0^{\frac{\mu}{3}} dx(x-\frac{\mu}{3})^2\Big]$$

$$= \frac{1}{\beta^3}[\frac{\pi^2}{3}(\frac{\mu}{3}) + \frac{1}{3}(\frac{\mu}{3})^3]$$

$$\langle B\rangle = \frac{2V}{3\beta^3}(\mu + \frac{1}{27\pi^2}\mu^3)$$

The above discussion, however, was naive in that we have ignored the role of leptons, ie this calculation does not conserve $B - L$. Therefore, we must modify the original partition function by including a chemical potential for leptons similar to that for baryons, and impose this constraint by including a Kronecker delta function:

$$Z = Tr[e^{-\beta H} e^{\mu_B \int d^4x q_i^\dagger q_i} e^{\mu_L \int d^4x l_i^\dagger l_i} \delta_{BL}]$$

where now

$$H = \sum_{i,k} |k|[a_k^{(i)\dagger} a_k^{(i)} + b_k^{(i)\dagger} b_k^{(i)} + c_k^{(i)\dagger} c_k^{(i)} + d_k^{(i)\dagger} d_k^{(i)}]$$

with the Baryon and Lepton operators expressed as

$$\int d^4x\, q^\dagger q = \frac{1}{3}\sum_k [a_k^\dagger a_k - b_k^\dagger b_k],$$

$$\int d^4x\, l^\dagger l = \sum_k [c_k^\dagger c_k - d_k^\dagger d_k].$$

The delta function can be represented in integral form as

$$\delta_{BL} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha e^{i\alpha(B-L)}$$

(We can choose any range covering a full period of 2π for the integration limits, but as we will find later, the range from $-\pi$ to π turns out to be particularly convenient.)

$$B - L = \sum_k (\frac{1}{3} a_k^{(i)\dagger} a_k^{(i)} - \frac{1}{3} b_k^{(i)\dagger} b_k^{(i)} - c_k^{(i)\dagger} c_k^{(i)} + d_k^{(i)\dagger} d_k^{(i)})$$

It follows that

$$Z = Tr\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha (\prod_{i,k} \exp((-\beta|k| + \frac{\mu_B}{3} + \frac{i\alpha}{3}) a_k^{(i)\dagger} a_k^{(i)}))(\prod_{i,k} \exp((-\beta|k| - \frac{\mu_B}{3} - \frac{i\alpha}{3}) b_k^{(i)\dagger} b_k^{(i)}))$$

$$\times(\prod_{i,k} \exp((-\beta|k| + \mu_L - i\alpha) c_k^{(i)\dagger} c_k^{(i)}))(\prod_{i,k} \exp((-\beta|k| - \mu_L + i\alpha) d_k^{(i)\dagger} d_k^{(i)}))\Big].$$

As before, the trace gives a sum over all possible states of the system (zero or one for fermions), thus:

$$Z = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha (\prod_{k,i} 1 + e^{(-\beta|k| + \frac{\mu_B}{3} + \frac{i\alpha}{3})})(\prod_{k,i} 1 + e^{(-\beta|k| - \frac{\mu_B}{3} - \frac{i\alpha}{3})})$$
$$\times(\prod_{k,i} 1 + e^{(-\beta|k| + \mu_L - i\alpha)})(\prod_{k,i} 1 + e^{(-\beta|k| - \mu_L + i\alpha)})$$

Unlike the case for baryons alone, we cannot exploit the fact that we wish to find the logarithm of Z to convert this product into a sum of logarithms because of the α integral. However, we can write

$$Z = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \exp\Big[V\int \frac{d^3k}{(2\pi)^3} 36[\ln(1 + e^{(-\beta|k| + \frac{\mu_B}{3} + \frac{i\alpha}{3})}) + \ln(1 + e^{(-\beta|k| - \frac{\mu_B}{3} - \frac{i\alpha}{3})})]$$
$$+9[\ln(1 + e^{(-\beta|k| + \mu_L - i\alpha)}) + \ln(1 + e^{(-\beta|k| - \mu_L + i\alpha)})]\Big]$$

Again we have taken k to the continuum limit. As before the baryons obtain a factor of 36 from the sum over i (6 flavors, 3 colors, 2 spin states), the leptons similarly obtain a factor of 3 generations $\times$ 3 spin states (two for each charged lepton and one each neutrino) $= 9$.

The integral in the exponent is the same integral we had in the purely baryonic case, but with the real parameter $\mu/3$ replaced by a complex parameter z, where $z = (\mu_B + i\alpha)/3$ for the baryonic terms, and $z = \mu_L - i\alpha$ for the leptonic terms. The integrand has poles along the imaginary axis of the z-plane at $\mathrm{Im}\{z\} = \pi + 2\pi n$, n integer, and they occur when $k = \mathrm{Re}\{z\}$. The analytic form of the integral we found for real z above,

$$\frac{1}{\beta^3}[\frac{\pi^2}{6}z^2 + \frac{1}{12}z^4 + \mathrm{const}]$$

is valid as long as $\mathrm{Im}\{z\}$ is restricted to the range $(-\pi, \pi)$. For $\mathrm{Im}\{z\}$ outside of this range, the function simply repeats with period 2π, which happens because the integrand was periodic in α. The closed form expression for the full periodic function can be written by replacing z with $\ln e^z$ in the polynomial. As long as we choose the range of integration over α to be from $-\pi$ to π, the original polynomial form above can be used. The partition function is then:

$$Z = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \exp[\frac{1}{2\pi^2\beta^3} f(\alpha, \mu_B, \mu_L) + \mathrm{const}]$$

where

$$f(\alpha, \mu_B, \mu_L) = 36[\frac{\pi^2}{6}((\mu_B + i\alpha)/3)^2 + \frac{1}{12}((\mu_B + i\alpha)/3)^4] + 9[\frac{\pi^2}{6}(\mu_L - i\alpha)^2 + \frac{1}{12}(\mu_L - i\alpha)^4]$$

The constant term is independent of both μ_B and μ_L. Since we are only interested in expectation values which do not depend on the normalization of the partition function, we can ignore this term.

The α integral can be performed using the saddle point approximation. Allowing α to be complex now, there is a maximum in $\mathrm{Re}\{f\}$ at $\mathrm{Re}\{\alpha\} = 0$. The saddle point occurs at the point on the imaginary axis where there is a minimum in $\mathrm{Re}\{f\}$ along the imaginary direction in the α-plane. In the text, we will find that $\mu_{L,B}$ must be small in order to fit the observed baryon asymmetry. Therefore, we will work to lowest order in the chemical potentials, and avoid writing the lengthy exact expression.

To first order in μ_L and μ_B, the saddle point is at:

$$\alpha_{\text{saddle}} = \frac{4i}{13}\mu_B - \frac{9i}{13}\mu_L$$

The value of f at α_{saddle}, to second order in μ_B and μ_L is:

$$f(\alpha_{\text{saddle}}, \mu_B, \mu_L) = 9.11\mu_B\mu_L + 4.55\mu_B^2 + 4.55\mu_L^2$$

Z can be approximated well by evaluating the integrand at α_{saddle}. In the saddle point approximation, this would usually need to be divided by the square root of the second derivative of f with respect to α, but this just leads to an additive term in $\ln Z$ which is independant of the volume of spacetime. As $V \to \infty$, this is negligible compared to the term proportionate to V.

$$\ln Z \sim \frac{V}{2\pi^2\beta^3} f(\alpha_{\text{saddle}}, \mu_B, \mu_L) = 0.46\frac{V}{\beta^3}(\mu_B\mu_L + \frac{1}{2}\mu_B^2 + \frac{1}{2}\mu_L^2) + O(\mu_{B,L}^3)$$

Therefore, the expectation values of baryon and lepton number are, to first order in μ_B and μ_L:

$$\langle B\rangle = \langle L\rangle = \frac{\partial \ln Z}{\partial \mu_B} = \frac{\partial \ln Z}{\partial \mu_L} \approx \frac{V}{2\beta^3}(\mu_B + \mu_L)$$

This calculation gives us the baryon asymmetry written in Equation 3.3, when we plug in the effective chemical potential in Equation 3.2.

References

[1] E. Witten, Phys. Rev. D **30**, 272 (1984). D. B. Kaplan, Phys. Rev. Lett. **68**, 741 (1992). F. Bauer, M. T. Eisele and M. Garny, Phys. Rev. D **74**, 023509 (2006) [arXiv:hep-ph/0510340].

[2] T. Banks, arXiv:hep-ph/0606313.

[3] A. G. Cohen and D. B. Kaplan, Nucl. Phys. B **308**, 913 (1988). A. G. Cohen and D. B. Kaplan, Phys. Lett. B **199**, 251 (1987).

[4] L. E. Ibanez, arXiv:hep-th/0602279.

[5] I. Affleck and M. Dine, Nucl. Phys. B **249**, 361 (1985). M. Dine, L. Randall and S. D. Thomas, Nucl. Phys. B **458**, 291 (1996) [arXiv:hep-ph/9507453].

[6] A.P. Zhitnitsky, *On the possible suppression of axion-hadron interactions*, Yad. Fiz. 31, 497, (1980) (Sov. J. Nuc. Phys. 31, 260, (1980); M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, *Can confinement ensure natural CP invariance of strong interactions?*, Nucl. Phys. B166, 493, (1980); M. Dine, W. Fischler and M. Srednicki, Phys. Lett. B **104**, 199 (1981). J. E. Kim, Phys. Rev. D **26**, 3221 (1982).

[7] T. Banks, M. Dine and M. Graesser, Phys. Rev. D **68**, 075011 (2003) [arXiv:hep-ph/0210256]. P. Fox, A. Pierce and S. D. Thomas, arXiv:hep-th/0409059.